Published by Institute of Physics Publishing for SISSA/ISAS
Received: *July 30, 2003*
Accepted: *September 18, 2003*

Renormalizability of non(anti)commutative gauge theories with $\mathcal{N} = 1/2$ supersymmetry*

Oleg Lunin[a] **and Soo-Jong Rey**[ab]

[a]*Institute for Advanced Study*
Einstein Drive, Princeton NJ 08540 USA
[b]*School of Physics & BK-21 Physics Division, Seoul National University*
Seoul 151-747 Korea
E-mail: lunin@ias.edu sjrey@gravity.snu.ac.kr

Abstract: Non(anti)commutative gauge theories are supersymmetric Yang-Mills and matter system defined on a deformed superspace whose coordinates obey non(anti)commutative algebra. We prove that these theories in four dimensions with $\mathcal{N} = 1/2$ supersymmetry are renormalizable to all orders in perturbation theory. Our proof is based on operator analysis and symmetry arguments. In a case when the Grassman-even coordinates are commutative, deformation induced by non(anti)commutativity of the Grassman-odd coordinates contains operators of dimension-four or higher. Nevertheless, they do not lead to power divergences in a loop diagram because of absence of operators hermitean-conjugate to them. In a case when the Grassman-even coordinates are noncommutative, the ultraviolet-infrared mixing makes the theory renormalizable by the planar diagrams, and the deformed operators are not renormalized at all. We also elucidate relation at quantum level between non(anti)commutative deformation and $\mathcal{N} = 1/2$ supersymmetry. We point out that the star product structure dictates a specific relation for renormalization among the deformed operators.

Keywords: Renormalization Regularization and Renormalons, Superstrings and Heterotic Strings, D-branes, Brane Dynamics in Gauge Theories.

*This work was supported in part by NSF grant PHY-0070928 (OL), the KOSEF Interdisciplinary Research Grant 98-07-02-07-01-5 (SJR), and the KOSEF Leading Scientist Grant (SJR).

JHEP09(2003)045

Contents

1. Introduction

Deformation of ordinary superspace has attracted renewed interest, sparked off in part by the study of string dynamics in the background of Ramond-Ramond flux [1]. A situation where this sort of issue is prominently brought up is string theoretic understanding of gauge theory - matrix model correspondence put forward by Dijkgraaf and Vafa [2] (See also [3] for other closely related motivations and interesting applications.).

Consider a D-brane in the background of Ramond-Ramond flux and open string dynamics on it. Much the same way the Kalb-Ramond 2-form potential B_{mn} affects algebra obeyed by Grassman-even coordinates [4], the Ramond-Ramond flux does so for the algebra of Grassman-odd coordinates [5]–[8]. For example, take a (space-filling) $D3$-brane of the Type IIB superstring theory compactified on a Calabi-Yau 3-fold X, where the Kalb-Ramond 2-form potential and the self-dual graviphoton flux are turned on along the flat spacetime, $\mathbb{R}^4$. If they are constant throughout $\mathbb{R}^4$, these background fields do not produce energy-momentum tensor and hence do not back-react to the geometry. Nevertheless, these background fields render nontrivial effect on the $D3$-brane worldvolume [7, 8]: $\mathcal{N} = 1$ supersymmetry is deformed to $\mathcal{N} = 1/2$ supersymmetry. In terms of chiral coordinates $z \equiv (y^m, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})$ ($m = 1, \ldots, 4, \alpha, \dot{\alpha} = 1, 2$) of the underlying $\mathcal{N} = 1$ superspace, the deformed superspace is such that

$$[y^m, y^n] = i\Theta^{mn} \qquad \text{and} \qquad \left\{\theta^\alpha, \theta^\beta\right\} = C^{\alpha\beta}\,, \tag{1.1}$$

where $\Theta^{[mn]}$, $C^{(\alpha\beta)}$ refer to combinations of background NS-NS (Neveu-Schwarz - Neveu-Schwarz) and R-R (Ramond-Ramond) fields. In a suitable low-energy decoupling limit ($\ell_{\rm st} \to 0$ and rescaling of the background fields), the open string dynamics on $D3$-brane worldvolume is described by Yang-Mills fields. With the background fields turned on, underlying $\mathcal{N}=1$ superspace is deformed to $\mathcal{N}=1/2$ superspace as in (1.1). Accordingly, underlying $\mathcal{N}=1$ supersymmetric gauge theory is deformed to $\mathcal{N}=1/2$ supersymmetric gauge theory, in which the ordinary product between (super)fields is replaced by the star product

$$\star = \exp\left(\frac{i}{2}\Theta^{mn}\overleftarrow{\frac{\partial}{\partial y^m}}\overrightarrow{\frac{\partial}{\partial y^n}} - \frac{1}{2}C^{\alpha\beta}\overleftarrow{\frac{\partial}{\partial\theta^\alpha}}\overrightarrow{\frac{\partial}{\partial\theta^\beta}}\right). \tag{1.2}$$

Thus, study of *open* string dynamics in the background Ramond-Ramond flux calls first for thorough understanding of the deformed supersymmetric gauge theory.

In this work, we take a step toward this direction.[1] We study quantum dynamics of $\mathcal{N}=1/2$ supersymmetric gauge theory and prove that the theory is renormalizable to all orders in perturbation theory. We also offer a deeper understanding concerning relations between the non(anti)commutative deformation and the $\mathcal{N}=1/2$ supersymmetry. We show that the non(anti)commutative deformation of $\mathcal{N}=1$ supersymmetric gauge theory gives rise to $\mathcal{N}=1/2$ one with a specific choice of coefficients for various deformed terms in the lagrangian. The choice is dictated by the star product (1.2). On the other hand, generic $\mathcal{N}=1/2$ supersymmetric gauge theories permit arbitrary coefficients and do not automatically bear the structure of the star product (1.2). Nevertheless, for arbitrary coefficients, we show that the theory is renormalizable.

In a related context, it was found recently that $\mathcal{N}=1/2$ Wess-Zumino model is renormalizable [12]. The model, however, was not the one obtained by non(anti)commutative deformation of underlying $\mathcal{N}=1$ Wess-Zumino model, but the one in which several operators (breaking the $\mathcal{N}=1$ supersymmetry to $\mathcal{N}=1/2$) were added by hand. In contrast, $\mathcal{N}=1/2$ supersymmetric gauge theory obtained via non(anti)commutative deformation is renormalizable by itself, and there is no need to add new operators as in the Wess-Zumino model case. This pleasant surprise arises because of several (pseudo)symmetries underlying the theory. More specifically, helicity conservation, R-symmetry, and flavor symmetries constrain possible ultraviolet-divergent operators and counterterms.

It is interesting that the theory is renormalizable even though it contains operators $\mathcal{O}$ of mass-dimension five or higher. Typically, a quantum field theory is defined by a hermitean self-conjugate lagrangian. In this case, one can show that insertion of such operators into a loop diagram renders the diagram power-divergent (thus the theory becomes nonrenormalizable), one can always connect an operator $\mathcal{O}$ with some diagrams, for example, by starting with a pair $\mathcal{O}\mathcal{O}^\dagger$ and using the vertices present in the theory. This constitutes the crux of power-counting in ordinary quantum field theories. On the other hand, in $\mathcal{N}=1/2$ theories obtained by the non(anti)commutative deformation (1.1), the hermiticity is broken as one deforms the chiral Grassman-odd coordinates θ^α but not the antichiral ones $\overline{\theta}^{\dot\alpha}$.

[1] Quantum dynamics of non(anti)commutative Wess-Zumino model was studied recently in [9]–[12].

JHEP09(2003)045

Therefore, higher-dimension operators are not accompanied by its hermitean-conjugates, and the ordinary power-counting argument would not apply. We will comment more on relation between lack of hermiticity and power-counting renormalizability in section 3.

This work is organized as follows. In section 2, we set up the $\mathcal{N}=1/2$ supersymmetric gauge theory with multi-flavor matter. We take the gauge group $G=\mathrm{U}(N)$. In section 3, by power-counting and symmetry arguments, we show that the theory is renormalizable. In section 4, we present operator analysis and classify requisite counterterms. We show that the theory contains only a finite number of operators receiving logarithmic divergences. In section 5, we consider turning on noncommutativity Θ^{ab} for Grassman-even coordinates, as in (1.1). Making use of known features concerning UV-IR (ultraviolet-infrared) mixing [13] (see [14] for a coherent account for the phenomenon), we show that the gauge theories are again renormalizable, but in a different sense that involves only the planar diagrams. We also discuss interplay between star product (1.2) and $\mathcal{N}=1/2$ supersymmetry. We assert that, in order for the quantum theory to retain the star product structure, radiative corrections for various deformed operators ought to obey certain scaling relations. We conclude in section 6 with discussions on issues worthy for further investigation.

2. $\mathcal{N}=1/2$ super Yang-Mills theory

The $\mathcal{N}=1/2$ supersymmetric gauge theory is defined as follows [7]. Start with real vector superfield V, valued in Lie algebra $g=$u(N), and define non(anti)commutative gauge transformation

$$e_\star^V \rightarrow e_\star^{V'} = e_\star^{-i\overline{\Lambda}} \star e_\star^V \star e_\star^{+i\Lambda}\,, \tag{2.1}$$

where the gauge transformation functions $\Lambda, \overline{\Lambda}$ are chiral and antichiral superfields. The chiral and antichiral field-strength superfields are defined by[2]

$$\begin{aligned} W_\alpha &\equiv -\frac{1}{4}\overline{DD}\left(e_\star^{-V} \star D_\alpha e_\star^V\right) \\ \overline{W}_{\dot\alpha} &\equiv +\frac{1}{4}DD\left(e_\star^{V} \star \overline{D}_{\dot\alpha} e_\star^{-V}\right). \end{aligned}$$

Under the non(anti)commutative gauge transformations (2.1), these field-strength superfields transform as

$$\begin{aligned} W_\alpha &\rightarrow W'_\alpha := e_\star^{-i\Lambda} \star W_\alpha \star e^{+i\Lambda}\,, \\ \overline{W}_{\dot\alpha} &\rightarrow \overline{W}'_{\dot\alpha} := e_\star^{-i\overline{\Lambda}} \star \overline{W}_{\dot\alpha} \star e_\star^{+i\overline{\Lambda}}\,. \end{aligned}$$

Fixing the gauge freedom by non(anti)commutative counterpart of the Wess-Zumino gauge, the vector superfield is reduced to

$$\begin{aligned} V(y,\theta,\overline{\theta}) \;=\; & -\theta\sigma^m\overline{\theta}A_m(y) + i\theta\theta\overline{\theta}\overline{\lambda}(y) - i\overline{\theta}\overline{\theta}\theta^\alpha\left(\lambda(y) + \frac{1}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^m_{\gamma\dot\gamma}\{\overline{\lambda}^{\dot\gamma}, A_m\}\right) + \\ & + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\left(D(y) - i\partial_m A^m(y)\right), \end{aligned}$$

where, following [7], $\overline{\theta}\overline{\theta}\theta$-term is modified so that the standard gauge transformation rule follows for component fields.

[2] We follow the convention of Wess and Bagger [15], but rescale the vector superfield as $V_{\mathrm{ours}} \leftrightarrow (2g)\,V_{\mathrm{WB}}$. Also we recall the relation between the conventional coupling constant g and the coupling constant e_{WB} of [15]: $e_{\mathrm{WB}} = 2g$.

JHEP09(2003)045

We also couple matter system by introducing a set of chiral superfields transforming in appropriate representations of the gauge group G. For example, a matter of N_F flavors with vectorlike coupling is described by $\Phi_f, \widetilde{\Phi}_{\widetilde{f}}$ $(f, \widetilde{f} = 1, \ldots, N_F)$ transforming in a pair of conjugate representations $\mathtt{R}$ and $\overline{\mathtt{R}}$:

$$
\begin{aligned}
\Phi_f(y,\theta) &= \varphi_f(y) + \sqrt{2}\theta\psi_f(y) + \theta\theta F_f(y) \\
\widetilde{\Phi}_{\widetilde{f}}(y,\theta) &= \widetilde{\varphi}_{\widetilde{f}}(y) + \sqrt{2}\theta\widetilde{\psi}_{\widetilde{f}}(y) + \theta\theta \widetilde{F}_{\widetilde{f}}(y) \\
\overline{\Phi}_f(\overline{y},\overline{\theta}) &= \overline{\varphi}_f(\overline{y}) + \sqrt{2}\overline{\theta}\overline{\psi}_f(\overline{y}) + \overline{\theta}\overline{\theta}\left(\overline{F}_f(\overline{y}) + iC^{mn}\partial_m(\overline{\varphi}_f A_n)(\overline{y}) - \frac{1}{4}C^{mn}\overline{\varphi}_f A_m A_n(\overline{y})\right) \\
\overline{\widetilde{\Phi}}_{\widetilde{f}}(\overline{y},\overline{\theta}) &= \overline{\widetilde{\varphi}}_{\widetilde{f}}(\overline{y}) + \sqrt{2}\overline{\theta}\overline{\widetilde{\psi}}_{\widetilde{f}}(\overline{y}) + \overline{\theta}\overline{\theta}\left(\overline{\widetilde{F}}_{\widetilde{f}}(\overline{y}) + iC^{mn}\partial_m(\overline{\widetilde{\varphi}}_{\widetilde{f}} A_n)(\overline{y}) - \frac{1}{4}C^{mn}\overline{\widetilde{\varphi}}_{\widetilde{f}} A_m A_n(\overline{y})\right),
\end{aligned}
$$

where $\overline{y}^m := y^m - 2i\theta\sigma^m\overline{\theta}$. Again, to ensure the standard gauge transformations of component fields, we have modified the $\overline{\theta}\overline{\theta}$-term [16]. Then, the $\mathcal{N} = 1/2$ supersymmetric gauge theory is described by the lagrangian with gauge coupling parameter $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$:

$$
\begin{aligned}
L = &\left[\overline{\Phi}_f \star e_\star^V \star \Phi_f + \overline{\widetilde{\Phi}}_{\widetilde{f}} \star e_\star^{-V} \star \widetilde{\Phi}_{\widetilde{f}}\right]_{\theta\theta\overline{\theta}\overline{\theta}} + W_\star(\Phi_f, \widetilde{\Phi}_{\widetilde{f}})\Big|_{\theta\theta} + \overline{W}_\star(\overline{\Phi}_f, \overline{\widetilde{\Phi}}_{\widetilde{f}})\Big|_{\overline{\theta}\overline{\theta}} + \\
&+ \mathrm{Tr}\left[\frac{i\tau}{16\pi}W^\alpha \star W_\alpha\right]_{\theta\theta} - \mathrm{Tr}\left[\frac{i\overline{\tau}}{16\pi}\overline{W}^{\dot{\alpha}} \star \overline{W}_{\dot{\alpha}}\right]_{\overline{\theta}\overline{\theta}}.
\end{aligned} \tag{2.2}
$$

We are mostly concerned with features arising from non(anti)commutativity of Grassman-odd coordinates, so we will set $\Theta^{mn} = 0$ in what follows. For simplicity, we will also set the coupling parameter $\tau = i4\pi/g^2$. Upon expanding the star product among the fields explicitly, the lagrangian (2.2) is decomposable into a sum of an ordinary part identical to the $\mathcal{N} = 1$ supersymmetric gauge theory:

$$
L_{\mathcal{N}=1} = \left[\overline{\Phi}_f e^V \Phi_f + \overline{\widetilde{\Phi}}_{\widetilde{f}} e^{-V}\widetilde{\Phi}_{\widetilde{f}}\right]_{\theta\theta\overline{\theta}\overline{\theta}} + \left[W(\Phi_f, \widetilde{\Phi}_{\widetilde{f}}) + \mathrm{Tr}\,\frac{i\tau}{16\pi}W^\alpha W_\alpha\right]_{\theta\theta} + (\mathrm{h.c.}),
$$

and a deformation part which depends on powers of the non(anti)commutativity parameter $C^{\alpha\beta}$. This deformation can be expressed as D-terms involving spurion fields [17]:

$$
\begin{aligned}
L_{\mathcal{N}=1/2} = &\left[\frac{1}{4g^2}\mathrm{Tr}\,U^{\alpha\beta}(D_\alpha W_\beta)\overline{W}^{\dot{\alpha}}\overline{W}_{\dot{\alpha}} - \frac{U^{\alpha\beta}}{2\sqrt{2}} \times \right.\\
&\times \left\{(D_\alpha \overline{D}_{\dot{\alpha}}\overline{\Phi}_f)\overline{W}^{\dot{\alpha}} D_\beta \Phi_f + (D_\alpha \overline{D}_{\dot{\alpha}}\overline{\widetilde{\Phi}}_{\widetilde{f}})\overline{W}^{\dot{\alpha}} D_\beta \widetilde{\Phi}_{\widetilde{f}}\right\} - \\
&- \frac{1}{2}U^{\alpha\beta}\left\{\overline{\Phi}_f D_\alpha W_\beta D^2\Phi_f + \overline{\widetilde{\Phi}}_{\widetilde{f}} D_\alpha W_\beta \widetilde{\Phi}_{\widetilde{f}}\right\} + \\
&\left. + \frac{U}{16g^2}\mathrm{Tr}(\overline{W}^{\dot{\alpha}}\overline{W}_{\dot{\alpha}})^2 + \frac{U}{16}\left\{\overline{\Phi}_f \overline{W}^2_{\dot{\alpha}} D^2 \Phi_f + \overline{\widetilde{\Phi}}_{\widetilde{f}}\overline{W}^2_{\dot{\alpha}} D^2 \widetilde{\Phi}_{\widetilde{f}}\right\}\right]_{\theta\theta\overline{\theta}\overline{\theta}},
\end{aligned}
$$

where the spurion fields are[3]

$$
U^{\alpha\beta} = C^{\alpha\beta}\theta\theta\overline{\theta}\overline{\theta} \qquad \text{and} \qquad U = |\,C|^2\theta\theta\overline{\theta}\overline{\theta}\,.
$$

[3]We denote $|C|^2 \equiv 4\det(C^{\alpha\beta}) = C^{mn}C_{mn}$.

For $W(\Phi,\widetilde{\Phi}) = \overline{W}(\overline{\Phi},\overline{\widetilde{\Phi}}) = 0$, which we are mostly concerned with in this work, the matter fields are massless. In this case, the theory has a chiral flavor symmetry:

$$\mathrm{U_A}(1): \qquad \Phi_f \to e^{i\gamma}\Phi_f, \qquad \widetilde{\Phi}_{\tilde{f}} \to e^{i\gamma}\widetilde{\Phi}_{\tilde{f}}\,. \tag{2.3}$$

In the next section, this symmetry will play a useful role for operator analysis and proof of renormalizability.

Under the $\mathcal{N}=1/2$ supersymmetry, component fields transform as

$$\begin{aligned} \delta A_m &= -i\overline{\lambda}\overline{\sigma}_m\epsilon \\ \delta\lambda_\alpha &= i\epsilon_\alpha D + (\sigma^{mn}\epsilon)_\alpha\left[F_{mn} + \frac{i}{2}C_{mn}\overline{\lambda}\overline{\lambda}\right]; \qquad \delta\overline{\lambda}_{\dot{\alpha}} = 0 \\ \delta D &= -\epsilon\sigma^m\nabla_m\overline{\lambda} \end{aligned} \tag{2.4}$$

for the vector superfield, as

$$\begin{array}{ll} \delta\varphi_f = \sqrt{2}\epsilon\psi_f & \delta\overline{\varphi}_f = 0 \\ \delta\psi^\alpha_f = \sqrt{2}\epsilon^\alpha F_f & \delta\overline{\psi}_{\dot{\alpha}\,f} = -i\sqrt{2}(\nabla_m\overline{\varphi}_f)(\epsilon\sigma^m)_{\dot{\alpha}} \\ \delta F_f = 0 & \delta\overline{F}_f = -i\sqrt{2}\nabla_m\overline{\psi}_f\overline{\sigma}^m\epsilon - i\overline{\varphi}_f\epsilon\lambda + C^{mn}\nabla_m(\overline{\varphi}_f\epsilon\sigma_n\overline{\lambda}) \end{array} \tag{2.5}$$

for the matter superfield $\Phi_f, \overline{\Phi}_f$, and similarly for $\widetilde{\Phi}_{\tilde{f}}, \overline{\widetilde{\Phi}}_{\tilde{f}}$.

The lagrangian (2.2) is given in terms of component fields [7, 16] as:[4]

$$\begin{aligned} L = {} & \frac{1}{2g^2}\,\mathrm{Tr}\left\{-\frac{1}{4}F_{mn}F^{mn} - i\overline{\lambda}\overline{\sigma}^m\nabla_m\lambda + \frac{1}{2}D^2\right\} + \\ & + \overline{F}_fF_f - i\overline{\psi}_f\overline{\sigma}^m\nabla_m\psi_f - \nabla_m\overline{\varphi}_f\nabla_m\varphi_f + \frac{1}{2}\overline{\varphi}_fD\varphi_f + \frac{i}{\sqrt{2}}(\overline{\varphi}_f\lambda\psi_f - \overline{\psi}_f\overline{\lambda}A_f) - \\ & - \frac{i}{4g^2}C^{mn}\,\mathrm{Tr}\left\{F_{mn}\overline{\lambda}\overline{\lambda}\right\} + \frac{|C|^2}{16g^2}\,\mathrm{Tr}(\overline{\lambda}\overline{\lambda})^2 - \frac{1}{\sqrt{2}}C^{\alpha\beta}(\nabla_m\overline{\varphi}_f)\sigma^m_{\alpha\dot{\alpha}}\overline{\lambda}^{\dot{\alpha}}\psi_{\beta\,f} + \\ & + \frac{i}{2}C^{mn}\overline{\varphi}_fF_{mn}F_f + \frac{|C|^2}{16}\overline{\varphi}_f\overline{\lambda}\overline{\lambda}F_f\,. \end{aligned} \tag{2.6}$$

To shorten the expressions, we suppressed terms involving the superfields $\widetilde{\Phi}, \overline{\widetilde{\Phi}}$ in the above lagrangian. These terms are obtainable by replacing $\Phi_f, \overline{\Phi}_f$ in (2.6) to $\widetilde{\Phi}_{\tilde{f}}, \overline{\widetilde{\Phi}}_{\tilde{f}}$.

Finally, in the normalization we adopt, the covariant derivative and field strength are defined by

$$\nabla_m\varphi = \partial_mA + \frac{i}{2}A_m\varphi, \qquad \nabla_m\lambda = \partial_m\lambda + \frac{i}{2}[A_m,\lambda], \qquad F_{mn} = \partial_mA_n - \partial_nA_m + \frac{i}{2}[A_m,A_n].$$

In the next section, we will prove that $\mathcal{N}=1/2$ supersymmetric gauge theory described by (2.6) is renormalizable to all orders in perturbation theory.

[4]The lagrangian is consistent with the normalization of Wess and Bagger after the vector superfield is rescaled as $V_{\mathrm{ours}} = (2g)V_{\mathrm{WB}}$.

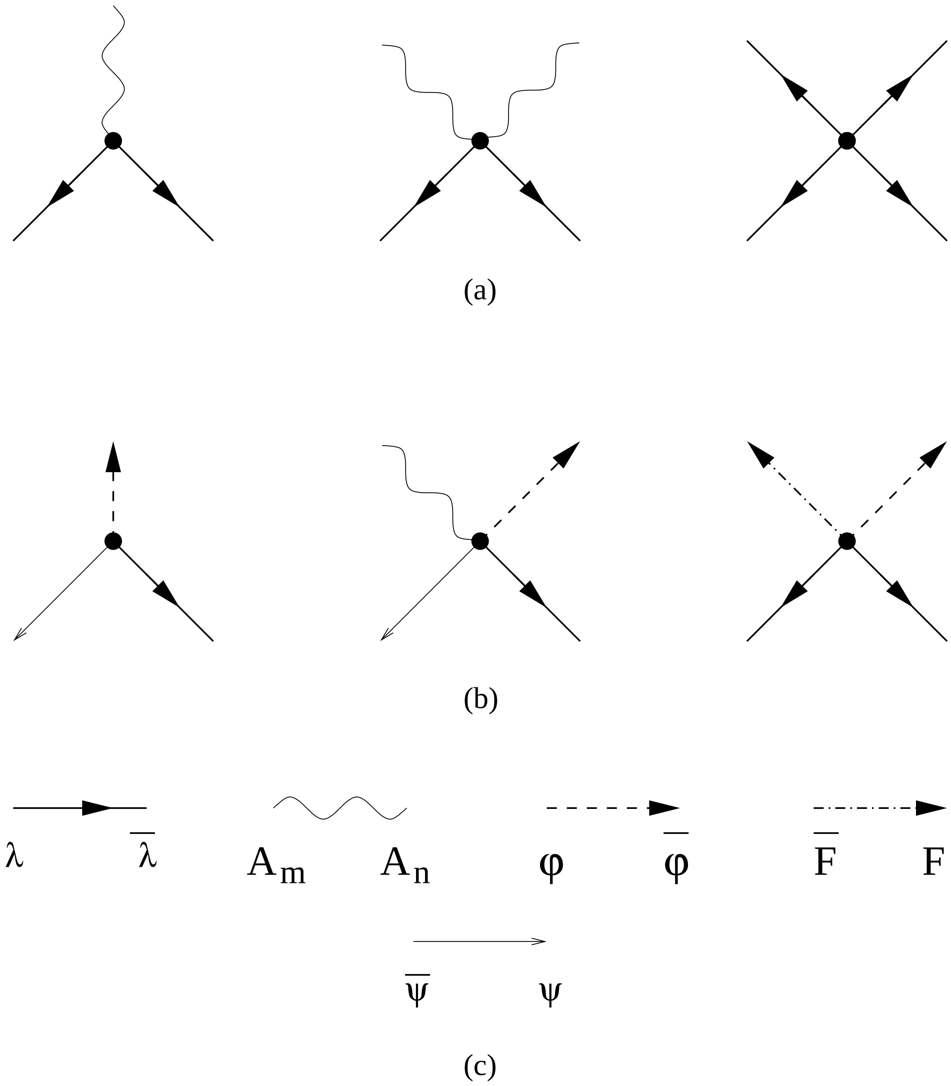

Figure 1: Deformed part $L_{\mathcal{N}=1/2}$ breaks the R-charge and the hermiticity. Vertices from (2.6) break R-charge by 2 or 4 : (a) deformation of the Yang-Mills coupling (first two terms in the last line of (2.6)); (b) deformation of the gauge–matter coupling (second two terms in the last line of (2.6). The R-charge flow is indicated by bold arrows. Propagators of Yang-Mills and matter component fields are summarized in (c).

3. Proof of renormalizability

To show that the theory (2.6) is renormalizable, we begin with the power-counting. Let us characterize ℓ–th line in a given Feynman loop diagram L by two numbers: r_ℓ which counts the power of momentum in the propagator ($r_\ell = -2$ for a boson and $r_\ell = -1$ for a fermion) and d_ℓ which counts the number of derivatives which act on a given propagator. For each vertex labeled by $i \in L$, we introduce an index ω_i:

$$\omega_i = \frac{1}{2}\sum_{\ell \in i}(r_\ell + d_\ell + 4) - 4\,,$$

where sum is performed over the propagators coming to the chosen vertex i. Then the 'superficial degree of divergence' for the loop diagram L is given by

$$\Omega_{\text{div.}}(L) = \sum_{i\in L}\omega_i - \left(\frac{1}{2}\sum_{\text{ext}}(r_\ell + d_\ell + 4) - 4\right), \tag{3.1}$$

where the last sum is performed only over the external lines. The Feynman loop diagram L is superficially divergent if $\Omega_{\text{div}}(L) \geq 0$.

We recall a quantum theory on $\mathbb{R}^4$ is referred renormalizable if it contains only a finite number of 'basic' diagrams with $\Omega_{\text{div}} \geq 0$. For ordinary (hermitean) theories in $\mathbb{R}^4$ renormalizability leads to the requirement that the theory contains interaction vertices with $\omega_i \leq 0$ only,[5] and contain only a finite number of 'basic' diagrams with $\Omega_{\text{div}} \geq 0$. The first two lines of (2.6) describe such renormalizable theory, and is in fact the $\mathcal{N} = 1$ supersymmetric gauge theory. On the other hand, the last line of (2.6) consist of operators with field-dimensions five or higher, but we will now show that the theory still contains a finite number of 'skeleton' divergent diagrams and thus it is renormalizable.

Before proceeding further, let us pause with recalling the standard argument that a field theory containing vertices with $\omega > 0$ is necessarily nonrenormalizable, as the argument fails for the class of theories under consideration. In the standard argument, one *assumes* that starting from any (convergent) loop diagram, one can add to it vertices with $\omega > 0$, thus making the diagram divergent. This assumption is justified by the hermiticity. For example, one can always add a vertex with $\omega > 0$ and its hermitean-conjugate. In a class of theories under consideration, the hermiticity is lost, as is evident from the observation that chiral Grassman-odd coordinates are deformed but not antichiral ones. The lack of hermiticity invalidates the conclusion drawn from the standard argument, and is ultimately responsible for renormalizability of the theory.

Thus, for the proof of renormalizability of (2.6), we will make extensive use of lack of hermiticity: many of the symmetries present in the undeformed $\mathcal{N} = 1$ theory are violated in the deformed $\mathcal{N} = 1/2$ theory. In keeping track of them, the most useful one is the (pseudo) R-symmetry:

$$\mathrm{U_R}(1): \qquad \varphi_f \to e^{-i\alpha}\varphi_f, \qquad F_f \to e^{+i\alpha}F_f, \qquad \lambda \to e^{-i\alpha}\lambda, \qquad C^{\alpha\beta} \to e^{-2i\alpha}C^{\alpha\beta} \tag{3.2}$$

and $\overline{F}$, $\bar{\varphi}$ and $\bar{\lambda}$ transforming with opposite charge and all the other fields being neutral. The lagrangian (2.6) is invariant under this pseudo R-symmetry, and there is only one coupling constant $C^{\alpha\beta}$ which is charged under it. The lack of hermiticity is reflected by the fact that there is no $C^{\dagger\dot{\alpha}\dot{\beta}}$ that can be assigned with the opposite R-charge. Stated differently, all the operators in $L_{\mathcal{N}=1/2}$ violate R-symmetry only by positive value of R-charges. This means that any loop diagram which contains vertices from the last line of (2.6) should have enough external lines to accommodate the conservation of the R-charge. As each vertex corresponding to an interaction with coupling constant $C^{\alpha\beta}$ or $|C|^2$ breaks the R symmetry (but it still preserves the pseudo R-symmetry (3.2)), so there

[5]This statement is equivalent to the absence of operators with field-dimensions higher then four.

ought to be some lines carrying positive R-charge which originate from such vertex. We illustrate this in figure 1. Since theory does not have vertices which decrease the R charge (by the lack of hermiticity), one can trace the "flow" of the positive R-charge all the way to the external lines. We refer the external states which lie on the 'lines of R-charge flow' as 'R-charge violating states', and it is convenient to separate them from the sum:

$$\begin{aligned}-\frac{1}{2}\sum_{\text{ext}}(r_\ell+d_\ell+4) &= -\frac{1}{2}\sum_{\text{R viol}}(r_\ell+d_\ell+4)-\frac{1}{2}{\sum_{\text{ext}}}'(r_\ell+d_\ell+4)\\ &\leq -N_R-\frac{1}{2}{\sum_{\text{ext}}}'(r_\ell+d_\ell+4).\end{aligned}$$

Here N_R is the number of the 'lines of R-charge flow', and the summation $\sum'$ does not include 'R-charge violating lines'. Since the 'lines of R-charge flow' can be traced back to the vertices from the third line of (2.6), for the superficial degree of divergence (3.1), we now get:

$$\Omega_{\text{div}} \leq \sum(\omega_i - N_R^{(i)}) + 4 - \frac{1}{2}{\sum_{\text{ext}}}'(r_\ell+d_\ell+4)\,.$$

Here, $N_R^{(i)}$ denotes the R-charge violated by a given vertex. We thus see that the index of the vertex ω_i is effectively replaced by $\hat{\omega}_i = \omega_i - N_R^{(i)}$ and the theory ought to be renormalizable if and only if this new index is non–positive for all vertices.

For the vertices originating from the undeformed part, viz. the ordinary $\mathcal{N}=1$ supersymmetric gauge theory, the new index $\hat{\omega}_i$ reduces to $\omega_i \leq 0$. For the vertices originating from the deformed part, viz. those $\mathcal{N}=\frac{1}{2}$ terms arising from non(anti)commutative deformation, we have:

$$\begin{array}{llll}\mathrm{Tr}\left\{\partial_{[m}A_{n]}\overline{\lambda\lambda}\right\}, & (\partial_m\overline{\varphi}_f)\overline{\lambda}^{\dot{\alpha}}\psi_{\beta\,f}, & \overline{\varphi}_f\partial_{[m}A_{n]}F_f: & \omega=\frac{1}{2},\quad N_R=2\,,\quad \hat{\omega}=-\frac{3}{2}\\ \mathrm{Tr}\left\{[A_mA_n]\overline{\lambda\lambda}\right\}, & (A_m\overline{\varphi}_f)\bar{\lambda}^{\dot{\alpha}}\psi_{\beta\,f}, & \overline{\varphi}_fA_{[m}A_{n]}F_f: & \omega=1\,,\quad N_R=2,\quad \hat{\omega}=-1\\ & \mathrm{Tr}(\overline{\lambda\lambda})^2\,, & \overline{\varphi}_f\bar{\lambda}\bar{\lambda}F_f: & \omega=2\,,\quad N_R=4\,,\quad \hat{\omega}=-2\,.\end{array} \tag{3.3}$$

We see that $\hat{\omega} \leq 0$ for all of these vertices. We have thus established the power-counting asserting that the non(anti)commutative $\mathcal{N}=1/2$ supersymmetric gauge theory is renormalizable. It now remains to classify all possible operators and counterterms that are subject to renormalization.

4. Operator analysis

To identify operators and counterterms needed for renormalization, we will utilize operator analysis and various symmetry arguments including the chiral flavor symmetry (2.3) and the pseudo R-symmetry (3.2). The most general, gauge-invariant, and $\mathcal{N}=1/2$ supersymmetric local operators that can appear through radiative corrections are expressible as[6]

$$\{\mathcal{O}_\Lambda\} = \Lambda^\beta\, C^\alpha_{mn}\, \partial^{\alpha_0}_m \cdot \varphi_f^{\alpha_1} \bar\varphi_f^{\bar\alpha_1}\, F_f^{\alpha_2} \bar F_f^{\bar\alpha_2}\, \psi_f^{\alpha_3} \bar\psi_f^{\bar\alpha_3} \cdot A_m^{\alpha_4} \lambda^{\alpha_5} \bar\lambda^{\bar\alpha_5}\, D^{\alpha_6}\,,$$

where Λ is an ultraviolet cutoff scale and α_i's are non-negative integers. Notice that, apart from the non(anti)commutativity parameter C^{mn}, there is no parameter in the theory with mass-dimension. This operator (including the coupling C^{mn}) should carry the mass-dimension four:

$$\beta - \alpha + \alpha_0 + \alpha_1 + \bar\alpha_1 + \alpha_4 + 2(\alpha_2 + \bar\alpha_2 + \alpha_6) + \frac{3}{2}(\alpha_3 + \bar\alpha_3 + \alpha_5 + \bar\alpha_5) = 4 \tag{4.1}$$

and the pseudo R-charge zero:

$$-2\alpha + \alpha_2 + \bar\alpha_1 + \bar\alpha_5 - \bar\alpha_2 - \alpha_1 - \alpha_5 = 0\,. \tag{4.2}$$

It follows from the last expression that

$$\bar\alpha_5 = 2\alpha + \bar\alpha_2 + \alpha_1 + \alpha_5 - \bar\alpha_1 - \alpha_2\,. \tag{4.3}$$

Then, (4.1) yields an equation for the remaining powers:

$$\beta + 2\alpha + \alpha_0 - \frac{1}{2}\bar\alpha_1 + \frac{5}{2}\alpha_1 + \alpha_4 + \frac{1}{2}\alpha_2 + \frac{7}{2}\bar\alpha_2 + 2\alpha_6 + 3\alpha_5 + \frac{3}{2}(\alpha_3 + \bar\alpha_3) = 4\,. \tag{4.4}$$

4.1 Class I. Operators without $\overline{\varphi}$

Equation (4.4) is useful for analyzing operators with $\bar\alpha_1 = 0$. We see that such operators have $\alpha \le 2$ and the only counterterm with $\alpha = 2$ has a schematic form:

$$(C_{mn})^2\overline{\lambda\lambda}\,\overline{\lambda\lambda}\,.$$

Analyzing indices more carefully, we find four possible types of operators with $\alpha = 2$:

$$\begin{aligned} \mathcal{O}_1 &= \det C\ \mathrm{Tr}(\overline{\lambda\lambda}\,\overline{\lambda\lambda})\,, & \mathcal{O}_2 &= |C|^2\ \mathrm{Tr}(\overline{\lambda}\overline{\sigma}_{mn}\overline{\lambda\lambda}\overline{\sigma}^{mn}\overline{\lambda})\,, \\ \mathcal{O}_3 &= C_{mn}C_{pq}\mathrm{Tr}(\overline{\lambda}\overline{\sigma}^{mn}\overline{\lambda\lambda}\overline{\sigma}^{pq}\overline{\lambda})\,, & \mathcal{O}_4 &= C_{mn}C_{pq}\mathrm{Tr}(\overline{\lambda}\overline{\sigma}^{mn}\overline{\lambda\lambda}\overline{\sigma}^{pq}\overline{\lambda})\,. \end{aligned} \tag{4.5}$$

Using Fierz identities of four-fermion and self–duality of C_{mn}, one can show that there is only one independent operator $\mathcal{O}_1$ and it is already present in the classical lagrangian (2.6).

Consider $\bar\alpha_1 = 0, \alpha = 1$. In order to be able to contract indices in $\mathcal{O}$, we should either have $\alpha_0 + \alpha_4 \ge 2$, or we should be able to introduce σ matrices into the vertex. We consider these two cases separately.

(*a*) For $\alpha_0 + \alpha_4 \ge 2$, (4.4) shows that the only nonvanishing powers are α_0, α_4. We also have $\bar\alpha_5 = 2$ and $\alpha_0 = 2 - \alpha_4$. So we have the following candidates:

$$C_{mn}\partial^2\bar\lambda\bar\lambda\,, \qquad C_{mn}\partial A_m\bar\lambda\bar\lambda, \qquad C_{mn}A_mA_n\overline{\lambda\lambda}\,.$$

[6]This expression should be understood as schematic: to simplify notation, we write component fields of Φ_f, but not those of $\tilde\Phi_{\tilde f}$. However, as they carry the same R-charge and mass-dimensions, the schematic notation does not affect our argument. To be more explicit, one should take, for example, $\phi_f^{\alpha_1}\tilde\phi_{\tilde f}^{\tilde\alpha_1}$ instead of $\phi_f^{\alpha_1}$ and use $\alpha_1 + \tilde\alpha_1$ wherever we have α_1.

JHEP09(2003)045

Using antisymmetry of C_{mn} and gauge invariance, we see that there is only one possible operator:

$$C_{mn}F^{mn}\bar{\lambda}\bar{\lambda}$$

and this is the operator already present in the classical lagrangian (2.6).

(*b*) For vertices with σ matrices, we should have at least two fermions. From (4.4) we see that $\alpha_5 = 0$ and we can not have more than one fermion. There are only the following operators:

$$C_{mn}\overline{\psi}_f\overline{\sigma}^{mn}\overline{\lambda}F_f\,, \qquad C_{mn}\overline{\tilde{\psi}}_{\tilde{f}}\overline{\sigma}^{mn}\overline{\lambda}\tilde{F}_{\tilde{f}}, \qquad C_{mn}\overline{\lambda}\overline{\sigma}^{mn}\overline{\lambda}\,,$$

but they vanish identically by virtue of self–duality of C^{mn}.

4.2 Class II. Operators involving $\overline{\varphi}$

For operators containing $\overline{\varphi}$ it is more convenient to extract $\bar{\alpha}_1$ from (4.2), then instead of (4.3) and (4.4) we get:

$$\bar{\alpha}_1 = \bar{\alpha}_2 + \alpha_1 + \alpha_5 + 2\alpha - \alpha_2 - \bar{\alpha}_5 \tag{4.6}$$

and

$$\beta + \alpha + \alpha_0 + \frac{1}{2}\bar{\alpha}_5 + 2\alpha_1 + \alpha_4 + \alpha_2 + 3\bar{\alpha}_2 + 2\alpha_6 + \frac{5}{2}\alpha_5 + \frac{3}{2}(\alpha_3 + \bar{\alpha}_3) = 4\,. \tag{4.7}$$

From (4.7) we see that there are no operators with $\alpha > 4$. Let us analyze operators with different $\alpha \leq 4$.

- For $\alpha = 4$ we get only one possible operator:

$$(C_{mn})^4(\overline{\varphi}_f\overline{\tilde{\varphi}}_{\tilde{f}})^4$$

  but it is inconsistent with the $\mathrm{U_A}(1)$ chiral flavor symmetry.

- For $\alpha = 3$ we need at least two indices from the operator to contract with the parameter C^{mn} (since $C^{mn}C^{np}C^{pm} = 0$), so we have two options:

  (*a*) For $\alpha_0 + \alpha_4 \geq 2$, (4.7) has no solution.

  (*b*) With at least two fermions, solving (4.7), we get an operator

$$(C^{mn})^3\overline{\lambda}\overline{\lambda}(\overline{\varphi}_f\overline{\tilde{\varphi}}_{\tilde{f}})^2\,,$$

  but it is not consistent with the $\mathrm{U_A}(1)$ chiral flavor symmetry.

- For $\alpha = 2$, it is convenient to classify the operators based on the number of fermions:

  (*a*) For the operators with at least one fermion from the matter multiplet, (4.7) gives only one operator:

$$(C^{mn})^2(\overline{\varphi}_f\overline{\tilde{\varphi}}_f)\overline{\varphi}_{f'}\overline{\lambda}\psi_{f'}\,,$$

  but again it is not consistent with the $\mathrm{U_A}(1)$ chiral flavor symmetry.

(b) Operators with four $\overline{\lambda}$ have $\overline{\alpha}_1 = 0$ and they are analyzed already.

(c) Operators with two $\overline{\lambda}$ must have $\alpha_2 = \overline{\alpha}_1$ to preserve the $\mathrm{U_A}(1)$ chiral flavor symmetry, so the only operator with nonzero $\bar{\alpha}_1$ is

$$(C^{mn})^2\overline{\varphi}_f\overline{\lambda\lambda}F_f\,, \qquad (C^{mn})^2\overline{\widetilde{\varphi}}_{\tilde{f}}\overline{\lambda\lambda}\widetilde{F}_{\tilde{f}}\,. \tag{4.8}$$

This operator is present already in the classical lagrangian (2.6). There are also other operators:

$$\Lambda(C^{mn})^2\overline{\varphi}_f\overline{\lambda\lambda\widetilde{\varphi}}_{\tilde{f}}, \qquad (C^{mn})^2\partial_k\overline{\varphi}_f\bar{\lambda}\bar{\lambda}\overline{\widetilde{\varphi}}_{\tilde{f}}, \qquad (C^{mn})^2A_k\overline{\varphi}_f\overline{\lambda\lambda\widetilde{\varphi}}_{\tilde{f}}. \tag{4.9}$$

These operators violate the chiral flavor $\mathrm{U_A}(1)$ symmetry and hence cannot appear for massless matter, but, the first operator with linear UV-divergence is potentially dangerous operator for the massive matter. We will analyze this case separately in the next subsection.

(d) For operators without fermions, we again have $\alpha_2 = \bar{\alpha}_1$, so the only operators with $\bar{\alpha}_1 \neq 0$ are

$$(C^{mn})^2(\overline{\varphi}_fF_f)^2\,, \qquad (C^{mn})^2(\overline{\varphi}_fF_f)(\overline{\widetilde{\varphi}}_{\tilde{f}}\widetilde{F}_{\tilde{f}}), \qquad (C^{mn})^2(\overline{\widetilde{\varphi}}_{\tilde{f}}\widetilde{F}_{\tilde{f}})^2\,. \tag{4.10}$$

These operators are consistent with all symmetries of the lagrangian, so they will be generated on quantum level. This means that one should introduce these terms even in the classical lagrangian to make the theory renormalizable.

- For $\alpha = 1$ and nonzero $\bar{\alpha}_1$, to be able to contract the indices of C^{mn}, we again should have one of the following situations:

(a) For $\alpha_0 + \alpha_4 \geq 2$, (4.7) shows that either $\alpha_0 + \alpha_4 = 3$, or $\alpha_0 + \alpha_4 = 2$, $\alpha_2 + \beta = 1$. In both cases $\bar{\alpha}_1 = 2 - \alpha_2$, so the chiral symmetry (2.3) requires $\bar{\alpha}_1 = \alpha_2 = 1$. This allows only one term consistent with gauge symmetry:

$$C^{mn}\overline{\varphi}_fF_{mn}F_f \tag{4.11}$$

which was already present in the lagrangian.

(b) For vertices with σ-matrices:

$$\begin{aligned} &C_{mn}\overline{\varphi}_f\psi_{\alpha\,f}(\sigma^{mn})_{\alpha\beta}\overline{\varphi}_{f'}\psi_{\beta,f'}\,, \qquad && C_{mn}A_m\overline{\varphi}_f\overline{\lambda}\sigma^n\psi_f\,, \\ &C_{mn}\partial_m\overline{\varphi}_f\overline{\lambda}\sigma^n\psi_f\,, && C_{mn}\overline{\varphi}_f\lambda\sigma\overline{\lambda}\,\overline{\widetilde{\varphi}}_{\tilde{f}}\,. \end{aligned} \tag{4.12}$$

Notice that the first vertex vanishes due to the Fermi statistics, and the last vertex cannot be contracted with C_{mn} (there is nothing like $(\sigma^{mn})_{\dot{\alpha}\beta}$). The other two vertices in (4.12) have to combine into a gauge-invariant operator

$$C^{\alpha\beta}\sigma^m_{\alpha\dot{\alpha}}\nabla_m\overline{\varphi}_f\bar{\lambda}^{\dot{\alpha}}\psi_{\beta\,f}\,,$$

but this operator is present already in the classical lagrangian (2.6). Equations (4.6), (4.7) have a solution for another vertex involving σ-matrices ($\bar{\alpha}_5 = 2, \bar{\alpha}_1 = 0$), but we already analyzed this type of operators above.

This concludes consideration of all possible operators with $\alpha > 0$, and the diagrams with $\alpha = 0$ have only the vertices of undeformed $\mathcal{N} = 1$ supersymmetric gauge theory, which is renormalizable by itself.

4.3 Massive matter

Recall that the theory (2.6) contains both Φ_f and it conjugate $\widetilde{\Phi}_{\tilde{f}}$. For massless matter, there is no coupling between Φ_f and $\widetilde{\Phi}_{\tilde{f}}$, so lagrangian is symmetric under chirality transformations (2.3). There is also the $\mathrm{U_F}(1)$ flavor symmetry:

$$\mathrm{U_F}(1): \qquad \Phi_f \to e^{i\delta}\Phi_f, \qquad \widetilde{\Phi}_{\tilde{f}} \to e^{-i\delta}\widetilde{\Phi}_{\tilde{f}}. \tag{4.13}$$

These two symmetries combined with the fact that all counterterms have no more than one power of Φ_f and $\widetilde{\Phi}_{\tilde{f}}$ (so that the operators like $(\overline{\Phi}_f\Phi_f)(\overline{\widetilde{\Phi}}_{\tilde{f}}\widetilde{\Phi}_{\tilde{f}})$ do not appear) ensure that the proof of renormalizability works separately for terms involving $(\varphi,\psi,F)_f$ and $(\widetilde{\varphi},\widetilde{\psi},\widetilde{F})_{\tilde{f}}$.

Our discussion so far was restricted to the case of massless theory (2.6). However, the main result (absence of counterterms which were not present in the original lagrangian) can be easily extended to the massive case as follows.

If mass is not equal to zero, then chiral symmetry is broken by the mass parameters:

$$L_m = m\left\{\varphi_f\widetilde{F}_{\tilde{f}} + \widetilde{\varphi}_{\tilde{f}}F_f - \psi_f\widetilde{\psi}_{\tilde{f}}\right\} + \overline{m}\left\{\overline{\varphi}_f\overline{\widetilde{F}}_{\tilde{f}} + \overline{\widetilde{\varphi}}_{\tilde{f}}\overline{F}_f - \overline{\psi}_f\overline{\widetilde{\psi}}_{\tilde{f}}\right\}. \tag{4.14}$$

Even though we ultimately want to consider real mass m, in $\mathcal{N}=1/2$ theory, m and $\bar{m}$ are independent coupling parameters. As such, the $\mathrm{U_A}(1)$ chiral flavor symmetry (2.3) is replaced for massive matter by a pseudo $\mathrm{U_A}(1)$ chiral-symmetry:

$$\mathrm{U_A}(1): \qquad \Phi \to e^{+i\gamma}\Phi, \qquad \tilde{\Phi} \to e^{+i\gamma}\tilde{\Phi}, \qquad m \to e^{-2i\gamma}m\,. \tag{4.15}$$

All possible counterterms in the theory must be invariant under the new pseudo $\mathrm{U_A}(1)$ chiral-symmetry.

In our discussion of massless case, we classified explicitly all possible operators and counterterms which were consistent with the pseudo R-symmetry, but not necessarily with the chiral flavor symmetry. We then argued that some of the terms were ruled out by the chiral symmetry. To discuss the massive theory, we just have to reconsider those operators and see whether some of them which were prohibited before by the chiral flavor symmetry might be consistent with the new pseudo $\mathrm{U_A}(1)$ chiral-symmetry.

We will use the chiral perturbation theory (i.e. perturbation theory in m and $\bar{m}$). Let us discuss various cases separately.

(a) If operator does not contain any power of m or $\overline{m}$, then it is invariant under chiral *symmetry* and such operators have been analyzed above.

(b) In the first order in m or $\overline{m}$, we have operators which break the chiral flavor symmetry by two units. On the other hand, since $m,\overline{m}$ carry dimension one, such operator can arise only from the one which diverges linearly with Λ, viz. by substituting Λ by $\overline{m}$. We had only one such operator: first term of (4.9), so we have two candidates:

$$\overline{m}|C|^2\overline{\widetilde{\varphi}}_{\tilde{f}}\overline{\lambda\lambda}\overline{\varphi}_f, \qquad \overline{m}|C|^2\overline{\varphi}_f\overline{\lambda\lambda}\overline{\widetilde{\varphi}}_{\tilde{f}}. \tag{4.16}$$

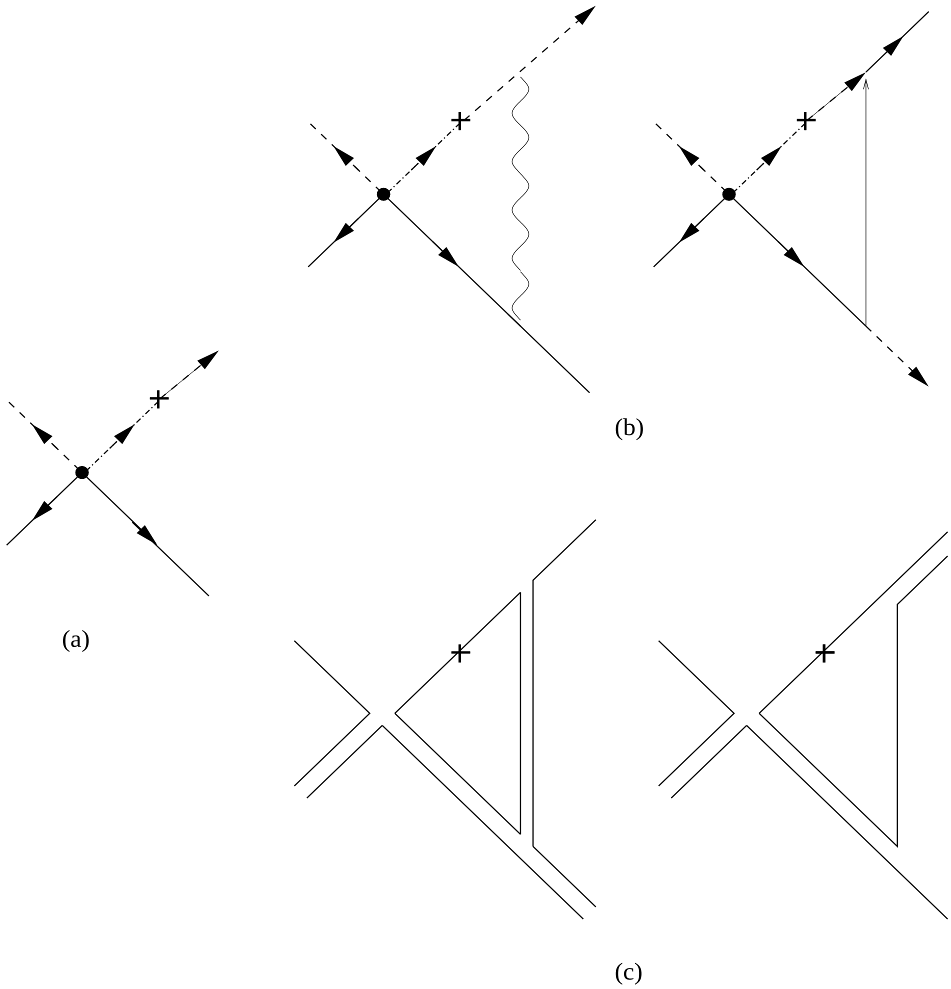

Figure 2: New vertex (4.16) arising in massive matter theory: (a) tree-level vertex, (b) one loop contribution to renormalization of the tree-level vertex in the usual notation; (c) the same diagrams in the double line notation. We see that one loop diagrams contribute at different orders in $1/N$. Cross indicates mass insertion.

These two operators are consistent with all symmetries of the lagrangian. In fact, they arise even at tree level in the chiral perturbation theory (see figure 2a). The tree-level vertex is radiatively corrected by the vertex renormalization in figure 2a as well as the mass renormalization. Moreover, there are additional renormalizations of the newly generated tree-level vertex. The one-loop diagrams which would contribute to such renormalization are presented in figure 2b. Each of these two diagrams yields a nonzero contribution to the renormalization of the vertex. The loop integral is the same in magnitude and opposite in sign, but the two cannot cancel each other as they have different orders in the color factor, N, as illustrated by 't Hooft double-line notation in figure 2c.

(c) To have operators with higher power of m and $\bar{m}$, we have to start from a counterterm which diverges as $\sim \Lambda^s$ with $s > 1$ in the massless theory. We found that the counterterms grows at most linearly with Λ (disregarding violation of the chiral flavor symmetry), so no operator with powerlike UV divergences are generated beyond first order in the chiral perturbation theory.

To summarize, we have analyzed all possible counterterms consistent with symmetries of deformed $\mathcal{N} = 1/2$ supersymmetric gauge theory, and we have shown that this theory is renormalizable by itself, viz. without introducing any new operators.

5. Star wars: confronting renormalizability and supersymmetry

So far, our proof for renormalizability was built largely on various symmetries in the theory, but not much directly on the $\mathcal{N} = 1/2$ supersymmetry or the non(anti)commutative star product. In this section, we elucidate relation between them and expose several intriguing features implicit to the operator analysis in the previous section.

- In the foregoing analysis, we have taken the noncommutativity parameter Θ^{mn} of the deformed superspace to zero. If we turn on Θ^{mn} nonzero and render the Grassman-even coordinates y^m noncommutative as well, an important modification arises regarding the radiative corrections. In noncommutative field theories – theories defined on spacetime with Θ^{mn}, an important feature of these theories was the UV-IR mixing [13]. Because of the noncommutativity, fields are representable as 't Hooft double lines and Feynman diagrams are classifiable into planar and nonplanar diagrams. The star product phase-factors all cancel out for planar diagrams but mixes external and loop momenta for nonplanar ones. Thus, any explicit noncommutativity parameter dependence would arise from nonplanar diagrams only. Put together, this leads to an important consequence that nonplanar diagrams are free from UV divergences (provided one keeps the external momenta nonzero).

  From (1.2), it is evident that the Grassman-even phase-factors are correlated with the Grassman-odd phase-factors. Therefore, planar or nonplanar diagrams classified according to the Grassman-even phase-factors would be the same as those with respect to the Grassman-odd phase-factors. This observation was made explicit for non(anti)commutative Wess-Zumino model in [11]. Now, combined with the consequence of UV-IR mixing for noncommutative field theories, this implies that terms depending explicitly on $C^{\alpha\beta}$ cannot have UV divergences, as they originate only from nonplanar diagrams — the otherwise UV divergences are transmuted via UV-IR mixing to IR divergences. We thus conclude that, in case Θ^{ab} nonzero, the theory is renormalized only through planar diagrams.

- Our proof of renormalizability does not depend on the $\mathcal{N} = 1/2$ supersymmetry. Notice that each of the three terms proportional to $C^{\alpha\beta}$ in the deformed lagrangian (2.6):

$$\frac{1}{g^2} C^{mn}\,\mathrm{Tr}\{F_{mn}\overline{\lambda\lambda}\}, \qquad C^{\alpha\beta}\nabla_m\overline{\varphi}_f\sigma^m_{\alpha\dot{\alpha}}\overline{\lambda}^{\dot{\alpha}}\psi_{\beta\,f} \qquad \text{and} \qquad C^{mn}\overline{\varphi}_f F_{mn} F_f \tag{5.1}$$

are related under supersymmetry transformations (2.4), (2.5) to the gaugino kinetic term and the matter $\overline{F}F$ term, respectively, but not to any other terms. Therefore, numerical coefficients in front of the terms (5.1) can be set arbitrary *provided* numerical coefficients of $C^{\alpha\beta}$-dependent terms in the supersymmetry transformations (2.4), (2.5) are adjusted accordingly. Notice also that each of the two terms proportional to $|C|^2$ in the deformed lagrangian (2.6):

$$(C^{mn})^2\,\mathrm{Tr}(\overline{\lambda\lambda}\,\overline{\lambda\lambda}) \qquad \text{and} \qquad (C^{mn})^2\overline{\varphi}_f(\overline{\lambda\lambda})F_f\,, \tag{5.2}$$

is invariant under the $\mathcal{N}=1/2$ supersymmetry transformations (2.4), (2.5) by itself. This indicates that the $\mathcal{N}=1/2$ supersymmetry alone would not fix the numerical coefficients of these terms and hence can be taken arbitrary values.

Our proof of renormalizability still holds without modification even if the numerical factors in front of these terms were chosen different from the ones in the lagrangian.

- In the lagrangian (2.6), numerical coefficients in front of the terms (5.1) and (5.2) are determined by the algebraic structure the star product (1.2). The two coefficients of terms (5.1) are correlated with the two coefficients of $C^{\alpha\beta}$-dependent supersymmetry transformations in (2.4), (2.5). They are thus fixable by normalization of $C^{\alpha\beta}$ in the definition of the star product (1.2). Then, the remaining two coefficients of terms (5.2) would reflect the algebraic structure of the star product, since they arise from second order upon expanding the exponential in (1.2). It implies that the star product representation for non(anti)commutativity is a structure imposed in addition to the $\mathcal{N}=1/2$ supersymmetry. Notice that the non(anti)commutativity deformation in (1.1):

$$\theta^\alpha\star\theta^\beta+\theta^\beta\star\theta^\alpha=C^{\alpha\beta}\,,$$

which leads immediately to the exponential form of the star product, is a structure a priori independent of supersymmetry.

The above consideration brings out the issue whether the star product (1.2) is stable against radiative corrections. The operator analysis in section 4 indicates that renormalization effects do not appear to relate or constrain size of radiative corrections to the deformation terms (5.1), (5.2). The renormalization factor Z_C for the non(anti)commutativity parameter $C^{\alpha\beta}$ (multiplied by the renormalization scale μ) is computable from radiative corrections to the terms (5.1). Similarly, the renormalization factor Z_{C^2} for the *square* of non(anti)commutativity parameter $|C|^2$ (multiplied by μ^2) is computable from radiative corrections to the terms (5.2). In general, these renormalization factors are related each other. For simplicity, consider the pure gauge theory with no matter. If the theory were to retain exponential structure of the star product (1.2), the two renormalization factors are constrained to be

$$(Z_C)^2=Z_{C^2}\,. \tag{5.3}$$

Though extremely interesting, in this work, we do not dwell on computations testing the relation (5.3), as dictated by the star product structure. We remark, however, that nonrenormalization of the star product is physically required, else the Grassman-even coordinates

JHEP09(2003)045

carry noncanonical dimension at quantum level. The nonrenormalizatoin is also necessary for the emergence of open Wilson lines. From the computation of effective superpotential in non(anti)commutative Wess-Zumino model, we expect that open Wilson lines again play important roles in non(anti)commutative gauge theories. Combined with gauge invariance, it is suggestive that the relation (5.3) would hold to all orders in perturbation theory.

6. Discussions

In this work, we have proven that the $\mathcal{N} = 1/2$ supersymmetric gauge theory is renormalizable to all orders in perturbation theory. Our proof is based largely on operator analysis and symmetry arguments. As such, it brings in many interesting questions worthy of further investigation.

- In proving the renormalizability, we have taken trivial superpotential for the matter superfields, either $W = \overline{W} = 0$ for massless case or $W = m\Phi_f\widetilde{\Phi}_{\widetilde{f}}, \overline{W} = \overline{m}\overline{\Phi}_f\overline{\widetilde{\Phi}}_{\widetilde{f}}$ for massive case. By combining the result of the present work and that of the Wess-Zumino model [12], we conjecture that non(anti)commutative Yang-Mills theory coupled to arbitrary matter contents and superpotential involving the most general, gauge-invariant quadratic and cubic interactions is renormalizable to all orders in perturbation theory. In this case, as in the case of the gauge theory with massive matter (discussed in the previous section) and of the Wess-Zumino model [9, 10, 12], operators that are not prohibited by underlying symmetry needs to be added. Part of our conjecture asserts that there are only finitely many such operators and that the newly added operators do not ruin renormalizability.

- The deformed gauge theory consists of two parts: ordinary part $L_{\mathcal{N}=1}$ with $\mathcal{N} = 1$ supersymmetry, and deformed part $L_{\mathcal{N}=1/2}$ with $\mathcal{N} = 1/2$ supersymmetry. We have shown that the ordinary part $L_{\mathcal{N}=1}$ receives radiative corrections only from itself and none from the deformed part $L_{\mathcal{N}=1/2}$. This implies that ordinary part of the wilsonian effective action is renormalized only at one-loop order. We also have shown that radiative corrections to the deformation part $L_{\mathcal{N}=1/2}$ arise only through interactions governed by the $L_{\mathcal{N}=1}$ part. Underlying to the renormalizability was violation of R-symmetry and helicity conservation and non-hermiticity of $L_{\mathcal{N}=1/2}$. It suggests that deformed part of the wilsonian effective action is renormalized at all orders in perturbation theory, as can be inferred from the operator relations concerning divergence of R-symmetry current, which is violated by the deformed part $L_{\mathcal{N}=1/2}$ as well as the anomaly. It would be interesting to analyze in detail renormalization group flow of the wilsonian effective action.

Acknowledgments

We acknowledge R. Britto, B. Feng, J. Maldacena and N. Seiberg for enlightening and useful discussions. S.J.R. was a Member at the Institute for Advanced Study during this work. He thanks the School of Natural Sciences for hospitality and for the grant-in-aid from the Fund for Natural Sciences.

JHEP09(2003)045

References

[1] See, for example, N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, *J. High Energy Phys.* **03** (1999) 018 [hep-th/9902098];
N. Berkovits, S. Gukov and B.C. Vallilo, *Superstrings in 2d backgrounds with RR flux and new extremal black holes*, *Nucl. Phys.* **B 614** (2001) 195 [hep-th/0107140].

[2] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[3] H. Kawai, T. Kuroki and T. Morita, *Dijkgraaf-Vafa theory as large-N reduction*, *Nucl. Phys.* **B 664** (2003) 185 [hep-th/0303210];
M. Hatsuda, S. Iso and H. Umetsu, *Noncommutative superspace, supermatrix and lowest Landau level*, hep-th/0306251;
Y. Shibusa and T. Tada, *Note on a fermionic solution of the matrix model and noncommutative superspace*, hep-th/0307236.

[4] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162];
N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[5] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, hep-th/0302078.

[6] H. Ooguri and C. Vafa, *The c-deformation of gluino and non-planar diagrams*, hep-th/0302109.

[7] N. Seiberg, *Noncommutative superspace, $N = 1/2$ supersymmetry, field theory and string theory*, *J. High Energy Phys.* **06** (2003) 010 [hep-th/0305248].

[8] N. Berkovits and N. Seiberg, *Superstrings in graviphoton background and $N = 1/2 + 3/2$ supersymmetry*, *J. High Energy Phys.* **07** (2003) 010 [hep-th/0306226].

[9] R. Britto, B. Feng and S.-J. Rey, *Deformed superspace, $N = 1/2$ supersymmetry and (non)renormalization theorems*, *J. High Energy Phys.* **07** (2003) 067 [hep-th/0306215].

[10] S. Terashima and J.-T. Yee, *Comments on noncommutative superspace*, hep-th/0306237.

[11] R. Britto, B. Feng and S.-J. Rey, *Non(anti)commutative superspace, UV/IR mixing and open Wilson lines*, *J. High Energy Phys.* **08** (2003) 001 [hep-th/0307091].

[12] M.T. Grisaru, S. Penati and A. Romagnoni, *Two-loop renormalization for nonanticommutative $N = 1/2$ supersymmetric WZ model*, *J. High Energy Phys.* **08** (2003) 003 [hep-th/0307099];
R. Britto and B. Feng, *$N = 1/2$ Wess-Zumino model is renormalizable*, hep-th/0307165;
A. Romagnoni, *Renormalizability of $N = 1/2$ Wess-Zumino model in superspace*, hep-th/0307209.

[13] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *J. High Energy Phys.* **02** (2000) 020 [hep-th/9912072];
M. Van Raamsdonk and N. Seiberg, *Comments on noncommutative perturbative dynamics*, *J. High Energy Phys.* **03** (2000) 035 [hep-th/0002186].

JHEP09(2003)045

[14] S.-J. Rey, *Exact answers to approximate questions: noncommutative dipoles, open Wilson lines and UV-IR duality*, hep-th/0207108;
J. Korean Phys. Soc. **39** (2001) S527;
Y. Kiem, S.J. Rey, H.T. Sato and J.T. Yee, *Open wilson lines as closed strings in noncommutative field theories*, *Eur. Phys. J.* **C 22** (2002) 781.

[15] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1992.

[16] T. Araki, K. Ito and A. Ohtsuka, *Supersymmetric gauge theories on noncommutative superspace*, hep-th/0307076.

[17] L. Girardello and M.T. Grisaru, *Soft breaking of supersymmetry*, *Nucl. Phys.* **B 194** (1982) 65.